Phenomenological modeling of the compaction dynamics of shaken granular systems

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Granular systems such as powder or sand can compact because of shaking. Using a phenomenological decay law for the successive inverse packing fractions, we obtain the relaxation dynamics for the packing fraction recently found in the experiment by Knight *et al.* [Phys. Rev. E **51**, 3957 (1995)] and discuss the physical implications as well as the physical origin of the proposed decay law. [S1063-651X(96)11608-1]

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I. INTRODUCTION

During the last decade, there has been an awakening of interest in the structural and dynamical behavior of granular systems (such as powder or dry sand) among physicists [1]. Granular systems are complex classical many-particle systems; their complexity results from the fact that the grains are finitely extended, can have complicated shapes, only interact through friction and inelastic collision, and compact due to gravity. Although many properties of granular systems such as heaping, avalanching, vibration-induced convection, and size segregation [1] are part of our daily experience, the physics behind these phenomena is just beginning to be understood. Another effect of wide practical applications in engineering and technology is the compaction of granular systems [2–6]. As an example, if you try to refill a pound of coffee powder in a jar made for a pound of coffee powder, you will usually figure out that this is not an easy task. While refilling, the coffee powder increases its volume and further compaction requires vertical shaking of the jar (with a closed lid). So far, compaction seems to be folklore: A static granular system can exist in a range of metastable states between the so-called loose- and close-packed limits. Shaking can compact the system. The question of how fast the compaction happens, however, is a highly nontrivial problem [2–6].

In a recent important work, Knight *et al.* [3,4] have explored the dynamics of the compaction process using well-determined experimental conditions: monodisperse spherical glass particles in a long thin vertical tube under the influence of a large number of vertical periodic shakes (typically 10^4) with a controlled shaking intensity Γ (determined by the ratio of the peak intensity of a shake and the gravitational acceleration). Above a critical value $\Gamma_c \simeq 1$, a peculiar dynamics for the successive compaction process from shake to shake happens. For $1.8 < \Gamma < 5$, Knight *et al.* [3,4] found that the most satisfactory fit of their data for the time dependence of (ensemble-averaged) packing fraction ρ_n at rest after successive shakes, $n = 1,2,3,\ldots$, has the functional form

$$\rho_n = \rho_\infty - \frac{\rho_\infty - \rho_0}{1 + B\ln(1 + n/\tau)}. \tag{1}$$

The parameters B and τ, as well as the initial and the final packing fraction ρ_0 (slightly larger than the loose-packed limit) and ρ_∞ (denoted ρ_f in [4]), can be extrapolated from the experiment [4]. As also shown by Knight *et al.* [4], B, τ, and ρ_∞ depend strongly on the applied shaking intensity Γ. The functional form (1) given by the authors without theoretical motivation [4] seems to be imcompatible with previous theoretical approaches on the compaction problem [5,6].

Compaction due to shaking is a property that does not appear in solids or Newtonian fluids and therefore, standard approaches using the equation of state fail. This is because granular systems show dynamical features that belong to the almost unexplored field of compressible viscoplastic ''fluids'' [7].

In this paper, we establish a possible scenario for the dynamics of granular compaction on a phenomenological level. We propose and provide reasons for a decay law for the successive inverse packing fractions from shake to shake. Its solution yields a time evolution of the packing fraction ρ_n which is — from the physicists point of view — equivalent to Eq. (1). Moreover, we explore the physics behind the coefficents B and τ in Eq. (1). Finally, we describe a simple time-continuous model for the decompaction-recompaction process during each shake that is based on viscoplastic arguments, in order to understand the physical origin of the proposed decay law.

II. THEORETICAL APPROACH

A. Order parameter and stroboscopic decay law

By bypassing, for the moment, the microscopic details of the decompaction and recompaction process that occur during each shake, we focus on finding the stroboscopic map (or the difference equation) that governs the dynamics leading to Eq. (1). As a convenient order parameter for the dynamic packing problem we introduce the *compaction ratio*

$$\alpha_n = \frac{\rho_n - \rho_\infty}{\rho_0 - \rho_\infty}, \tag{2}$$

which is the difference of the packing fraction ρ_n at rest after the shake n from its closed-packed limit ρ_∞ reduced by its initial difference and, therefore, it is an *experimentally measurable* quantity. Note that (i) using α_n instead of ρ_n allows us to eliminate the fitting parameters, ρ_0 and ρ_∞, from the

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subsequent discussion, (ii) α_n is positive semidefinite for all n, (iii) the initial compaction ratio α_0 equals unity, and (iv) α_n approaches zero in the limit $n\to\infty$. Therefore, the initial packing fraction and the close-packed limit of a granular system correspond to compaction ratios unity or zero, respectively. The question that will be addressed is the mechanism that leads to the empirical formula of Knight *et al.* [3,4].

Staying within the framework of discrete dynamical systems, the structure of a one-dimensional stroboscopic map that can lead to the compaction dynamics is

$$\alpha_n = f(\alpha_{n-1},n) = \frac{\alpha_{n-1}}{1+h_n\alpha_{n-1}} \tag{3}$$

with a coefficient $h_n>0$, which can be time dependent. Equation (3) fullfils the limits $\alpha_0=1$ and $\alpha_\infty=0$ as well as the decay to zero if h_n is bounded from above. The limit $h_n=0$ for all n corresponds to the solid limit without any compaction taking place, $\alpha_n=\alpha_0=1$ for all n. Supposing that h_n equals a constant c being *independent* of n leads to an algebraic decay, $\alpha_n=1/(1+cn)$. Therefore, to obtain a slower decay as found by Knight *et al.* [3,4], h_n in Eq. (3) has to be time dependent and decay weakening. This implies that h_n must decrease in time. As one can show, a purely algebraic decay, $h_n\sim 1/n$, is not adequate to reproduce Eq. (1). An ansatz of h_n, however, that allows for different decays for short and long times,

$$h_n = \frac{C}{1+n/\nu}, \tag{4}$$

is the simplest appropriate form of h_n that can reproduce the empirical result, Eq. (1), as we show below. Here, the parameter C and the characteristic decay time ν do not depend on the time n, but can depend — at least in general — on the vibration intensity Γ of the shaking process and on structural properties of a granular system such as grain size, grain material, grain shape, etc. One immediately infers that C and ν must be positive in order to guarantee an increase of the packing fraction ρ_n in time.

The nonlinear, nonautonomous map (3) for the compaction ratio α_n with h_n from (4) can be transformed into a *linear* nonautonomous map for the inverse compaction ratio $1/\alpha_n$,

$$\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}} = \frac{C}{1+n/\nu}, \quad n=1,2,3,\ldots. \tag{5}$$

The *stroboscopic decay law*, Eq. (5), is our central model for the compaction process. It allows a simple phenomenological interpretation of the dynamics of the packing process generated by *periodic* shaking: The difference of sucessive reciprocal compaction ratios decays (i) linearly in time n for times that are short in comparison to a characteristic decay time, and (ii) proportional to the inverse of time, $1/n$, in the long time limit. Finally, Eq. (5) also allows easily an experimental verification.

B. Exact solution of Eq. (5)

Equation (5) can be solved recursively by taking advantage of the properties of harmonic series [8,9]. The result is the following.

Proposition: Supposing the stroboscopic decay law (5) holds for periodic shaking with C and ν positive, the compaction ratio α_n after the nth shake reads

$$\alpha_n = \frac{1}{1+C\nu[\Psi(n+1+\nu)-\Psi(1+\nu)]}, \tag{6}$$

$n=0,1,2,\ldots,\infty$, with $\Psi(x)$ denoting the digamma function [8,9] defined as the logarithmic derivative of the gamma function.

Proof: In order to derive Eq. (6), we use the fact that the process starts at the initial reciprocal compaction ratio $1/\alpha_0\equiv 1$ at time $n=0$, and obtain using standard summation methods of calculus

$$\begin{aligned}\frac{1}{\alpha_n} &= \frac{1}{\alpha_{n-1}}+\frac{C\nu}{n+\nu} = 1+C\nu\sum_{k=1}^{n}\frac{1}{k+\nu}\\ &= 1+C\nu\left(\sum_{k=1}^{\infty}\frac{1}{k+\nu}-\sum_{k=1}^{\infty}\frac{1}{k+n+\nu}\right)\\ &= 1+C\nu\left[\sum_{k=1}^{\infty}\left(\frac{1}{k+\nu}-\frac{1}{k}\right)-\sum_{k=1}^{\infty}\left(\frac{1}{k+n+\nu}-\frac{1}{k}\right)\right]\\ &= 1-C\nu[\Psi(1+\nu)-\Psi(n+1+\nu)].\end{aligned} \tag{7}$$

In the last line of Eq. (7), we have introduced the digamma function $\Psi(x)$, which possesses the series expansion [8]

$$\Psi(x) = -\sum_{k=0}^{\infty}\left(\frac{1}{k+x}-\frac{1}{k+1}\right)-\gamma \tag{8}$$

with $\gamma=0.57721,\ldots$ denoting the Euler-Mascheroni constant. Expressing Eq. (7) in terms of α_n yields Eq. (6).

Several remarks are in order. (i) Equation (6) represents the unique solution of the decay law (5), and therefore, also the unique solution of the nonlinear map (3) and (4). (ii) α_n only depends on ν, the product of C and ν, and the shake number n. In order to have compaction at all, it is necessary that C and ν are both nonzero; otherwise, $\alpha_n=\alpha_0=1$ for all n. (iii) Since $\Psi(x)\to\infty$ as $x\to\infty$, α_∞ equals zero. (iv) Since $\Psi(x)$ is a monotonically increasing function for $x>0$, α_n decreases monotonically to zero. (v) For times n being large in comparison to $1+\nu$, the denominator of (6) increases logarithmically in time to infinity, $1/\alpha_n\sim\ln(n)+1-\Psi(1+\nu)+O(1/n;\nu/n)$. This implies that for large times n, the compaction ratio α_n approaches zero as $1/(C\nu\ln n)$, being very slow in comparison to an algebraic or exponential decay in time. (vi) The bracketed term on the right-hand side of Eq. (7) can be written as $\Sigma_{l=1}^{\infty}\Psi(l,1+\nu)n^l$ with $\Psi(l,x)=(d^l/dx^l)\Psi(x)$ denoting the polygamma functions. For short times, $n/(1+\nu)\ll 1$, one obtains $\alpha_n\simeq 1/[1+C\nu\Psi(1,1+\nu)n]$ with $\nu\Psi(1,1+\nu)\simeq 1$ for $\nu>0$. This implies a linear decrease of α_n with n for short times. (vii) The solution (6) also holds for negative C and/or ν. $C<0$ and $\nu>0$, however, leads to a slow increase

of α_n with n with a divergence at a time n when the denominator in (6) approaches zero. If the nondivergent initial part of this solution is also applicable to the shake-to-shake decompaction, behavior seen for very large Γ [10] remains (due to the lack of quantitative measurements) open.

C. Equivalence with the experimental result?

At first glance, it is not obvious that Eq. (6) reproduces the experimental result of Knight *et al.* [3,4]. In order to compare Eq. (6) with Eq. (1), we rewrite (6) in terms of the packing fraction ρ_n using Eq. (2), $\rho_n=\rho_\infty+(\rho_0-\rho_\infty)\alpha_n$, and obtain

$$\rho_n=\rho_\infty+\frac{\rho_0-\rho_\infty}{1+C\nu[\Psi(n+1+\nu)-\Psi(1+\nu)]}. \tag{9}$$

Since the digamma function $\Psi(x)$ diverges to ∞ as $x\to\infty$, ρ_n in (9) reproduces the limits ρ_0 and ρ_∞ for $n=0$ and $n\to\infty$, respectively. We need to show that the relaxation for the packing fraction, Eq. (9), reproduces the extrapolation formula of Knight *et al.* [3,4]. Using the asymptotics of the digamma function for large n and matching that with the initial value, one obtains for the second term in the denominator of Eq. (9)

$$\begin{aligned}\Psi(n+1+\nu)-\Psi(1+\nu)&=\ln n-\Psi(1+\nu)+O\left(\frac{1}{n};\frac{\nu}{n}\right)\\&=\ln[n\exp\{-\Psi(1+\nu)\}]+O\left(\frac{1}{n};\frac{\nu}{n}\right)\\&=\ln\left[1+\frac{n}{n_c}\right]+O\left(\frac{1}{n};\frac{\nu}{n}\right),\end{aligned} \tag{10}$$

where the characteristic relaxation time n_c is given by $n_c=\exp\{\Psi(1+\nu)\}$. Although Eq. (10) is only an asymptotic result (that, as a matter of fact, also fulfills the $n=0$ limit), a numerical comparison of the left- and right-hand sides of Eq. (10) shows a very high accuracy for the relevant parameter ranges of the decay rate ν (cf. the discussion in Sec. II D) even for short times. In Fig. 1, we plot the ratio

$$R=\frac{\Psi(n+1+\nu)-\Psi(1+\nu)}{\ln(1+n/n_c)} \tag{11}$$

for (a) $\nu=1.3$ and (b) $\nu=10^4$ and $n=1$–50. R determines the relative size of the corrections $O(1/n;\nu/n)$ in Eq. (10) in comparison to unity. As one can see from Fig. 1, there is no difference from zero within the linewidth for the small decay rate $\nu=1.3$; for the large decay rate, $\nu=10^4$, the maximum error (for $n=1$) is only about half of a percent. Insertion of Eq. (10) into Eq. (9) shows that Eq. (9) possesses basically the same functional structure as the extrapolation of Knight *et al.* [3,4], Eq. (1). This allows us to relate the coefficients C and ν in Eq. (5) to the coefficients B and τ measured by Knight *et al.* [3,4], yielding

$$\tau=n_c=\exp\{\Psi(1+\nu)\}, \tag{12}$$

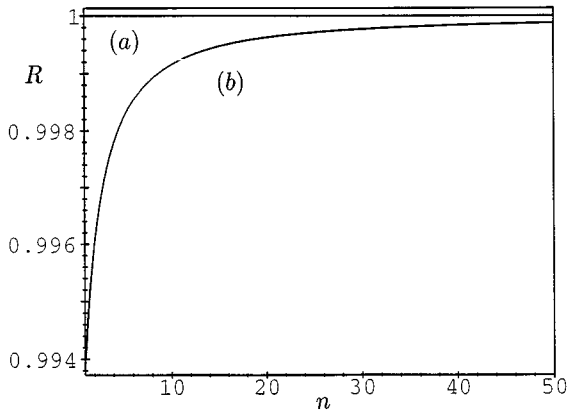

FIG. 1. Ratio R, Eq. (11), as a function of time $n=1$–50 for (a) $\nu=1.3$ and (b) $\nu=10\,000$.

$$B=C\nu. \tag{13}$$

Since C and ν are positive in our theory, B as well as τ are also positive in accordance with the experimental findings [4].

To summarize Secs. II A–II C, we conclude that the dynamics of the packing fraction, Eq. (9), derived from the stroboscopic decay law (5), is — from the physicist's point of view— equivalent to the empirical formula of Knight *et al.* [3,4] for the relevant ranges of the parameter ν. This in turn is a strong indication for the validity of the stroboscopic decay law (5) for the compaction process.

D. Role of the decay rate ν and the parameter C

So far, our theoretical approach does not specify any dependence on the shaking intensity Γ; it is hidden in the parameters C and ν. From their experimental data, Knight *et al.* [4] have determined the dependence of $\ln B$ and $\ln\tau$ as a function of the vibration intensity Γ in the range $1<\Gamma<5$. They found that both quantities decay rapidly from very large values $B\sim O(10^3)$ and $\tau\sim O(10^5)$ at $\Gamma\simeq 1$ to comparably small values $B\simeq 10^{-1}$ and $\tau\simeq 1.8$ at $\Gamma\simeq 3$. For larger Γ (at least up to $\Gamma\simeq 5$), B and τ are basically constant. For the τ values relevant in the Knight *et al.* experiment [4], τ can be approximated with high accuracy by

$$\tau\simeq\nu+\tfrac{1}{2} \tag{14}$$

since an expansion of Eq. (12) in terms of $1/\nu$ yields a rapidly converging series,

$$\tau=\nu+\frac{1}{2}+\frac{1}{24\nu}-\frac{1}{48\nu^2}+\frac{23}{5760\nu^3}+\frac{17}{3840\nu^4}+O\left(\frac{1}{\nu^5}\right). \tag{15}$$

Even for the smallest τ value found in the experiment [4], $\tau\simeq 1.8$, one obtains from Eq. (14) $\nu=1.3$, which is in good agreement with the exact result $\nu=1.28$. Therefore, we conclude that the characteristic decay time ν of successive reciprocal compaction ratios in Eq. (5) is basically proportional to the relaxation time τ of the packing fraction. Also the magnitude of the parameter $C=B/\nu$ in Eq. (5) can be estimated based on the results of Knight *et al.* [4]; the result is

that C is typically a quantity of order 10^{-2} for the whole range of relevant vibration intensities, $1.8<\Gamma<5$.

A further interesting result can be obtained by relating the coefficients B and τ of Knight *et al.* [4]. Combining Eqs. (12), (13), and (14), leads to a relation between $\ln B$ and $\ln\tau$:

$$\ln\tau \simeq \ln\left(\nu+\frac{1}{2}\right)=\ln B+\ln\left[\left(1+\frac{1}{2\nu}\right)\frac{1}{C}\right]. \tag{16}$$

In Fig. 5 of their paper, Knight *et al.* [4] show the dependence of $\ln B$ and $\ln\tau$ on the vibration intensity Γ. Using their data for $\Gamma\geqslant 1.8$, we suspect that a relationship $\ln\tau=\ln B+\ln K$ (or $\tau=KB$) with $K\simeq 18$ being independent of the vibration intensity Γ, is compatible with their experiment. From a comparison with Eq. (16), one can then conjecture that the parameter C and the decay rate ν are related by

$$C=\frac{1}{K}\left(1+\frac{1}{2\nu}\right), \tag{17}$$

with K being a constant that can depend on structural properties of the granular system, but is *independent* of the vibration intensity Γ. In particular for large enough decay rates ν, $C\simeq 1/K$, implying that C is also practically independent of the vibration intensity.

Below the threshold shaking intensity $\Gamma_c\simeq 1$ successive compaction does not take place, and the granular system behaves like a solid body. This results from the viscoplastic yield of the granular system and is reflected in Eq. (5) by a sudden jump of ν and/or C to zero for $\Gamma\leqslant\Gamma_c$.

E. Physical implications of Eq. (5)

Based on the experimental findings of Knight *et al.* [3,4] and our model, we can offer a *macroscopic* interpretation of the compaction dynamics under periodic shaking. Between the random loose-packed limit or an initial packing fraction ρ_0 (or $\alpha_0=1$) slightly larger than the loose-packed limit and the random close-packed limit ρ_∞ (or $\alpha_\infty=0$), a periodically shaken granular system with a large enough grain size to system size ratio goes through an infinite sequence of metastable packing states or compaction ratios α_n until the maximum compacted state $\alpha=0$ has been reached. Starting with $\alpha_0=1$, the decay of the compaction ratio due to periodic shaking happens—at least in principle—on two distinct time regimes with different decay behavior reflecting the two contributions in Eq. (5), the linear decrease in n for short times and the $1/n$ decrease for long times. For short times $n/\nu\ll 1$, the decay law (5) can be approximated by $1/\alpha_n-1/\alpha_{n-1}\simeq C(1-n/\nu)$, leading to a short-time compaction dynamics

$$\alpha_n\sim 1-Cn, \quad n/\nu\ll 1, \tag{18}$$

being independent of ν at this order, whereas the long-time dynamics is governed by

$$\alpha_n\sim\frac{1}{C\nu\ln n}, \quad n/\nu\gg 1. \tag{19}$$

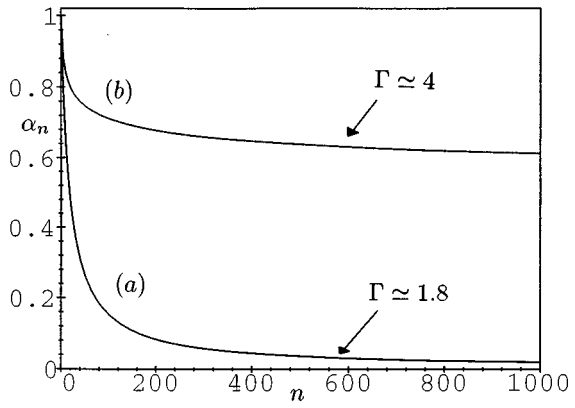

FIG. 2. Dependence of α_n on the time n, Eq. (7). The curve (a) with $\nu=10\,000$ and $C=0.056$ corresponds to a vibration intensity $\Gamma=1.8$, the curve (b) with $\nu=1.3$ and $C=0.077$ to $\Gamma=4$. These values for C and ν are extrapolations from the experimental results for B and τ in Ref. [4].

This implies (i) a fast relaxation of the compaction ratio decaying linearly with n for short times and (ii) a slow $1/\ln n$ relaxation for longer times. The crossover between these two time ranges occurs typically at about $n/\nu\sim O(0.1)$. Consequently, for values of ν of unit order (or vibration intensities Γ larger than 3), the crossover happens immediately at the beginning and the relaxation dynamics shows basically the $1/\ln n$ behavior. For large ν (vibration intensities close to $\Gamma\simeq 2$), the crossover occurs at times of the order 10^2 to 10^3 and both relaxation dynamics can be observed. In Fig. 2, we demonstrate this effect by showing the time evolution of the compaction ratio α_n, Eq. (6), for a large and a small vibration intensity Γ. These interpretations seem to be compatible with the experimental data of Knight *et al.* [4].

F. Physical origin of the stroboscopic law, Eq. (5)

So far, we have shown that the stroboscopic decay law, Eq. (5), leads to the compaction dynamics found by Knight *et al.* [3,4]. But what is the physical origin of Eq. (5) and, in particular, where does the inhomogeneity in Eq. (3) or Eq. (5) come from? Let us now sketch a simple, qualitative, exactly solvable model that is based on viscoplastic arguments [11] and mimics the decompaction and recompaction processes that occur from shake to shake. As a starting point, we introduce the time-continuous version of the compaction ratio, given by

$$\alpha(t)=\frac{\rho(t)-\rho_\infty}{\rho_0-\rho_\infty}, \tag{20}$$

which is positive and can reach values larger than unity during the decompaction-compaction process. Next, we make some simplifying assumptions: (i) the changes of the packing fraction during decompaction and recompaction are basically homogeneous along the height of the system; (ii) there is negligible friction at the side walls, and (iii) there is no liftoff of the grains at the bottom of the container while decompacting [12]. Using the first assumption, that $\alpha(t)$ is basically proportional to the rescaled and nondimensionalized height of the granular system, one can model its dynamics using

Newton's equation. In the following, all quantities are considered to be nondimensionalized by appropriately chosen length and time scales.

Suppose the granular system is at rest after the $(n-1)$th shake and possesses a compaction ratio α_{n-1}. At time $t=n$ the nth shake occurs. Viscoplastic yield requires that decompaction from rest only occurs if the shake intensity overcomes a certain value. In the experiment [4], the excitation of decompaction has a complicated pulse profile. This can be mimicked by a decompaction rate $v_n=\dot{\alpha}(n)$, which is the initial condition that starts the nth decompaction process and overcomes a yield decompaction rate $v_c=\dot{\alpha}_c>0$. Since the grains are not elastically coupled, there is no restoring force except the downwards acting gravity. Therefore, the decompaction process $[\dot{\alpha}(t)>0]$ can be modeled by

$$\ddot{\alpha}=-kg\Theta(v_n-v_c) \quad \text{if } \dot{\alpha}>0, \tag{21}$$

with $\Theta(x)$ denoting the Heaviside function, $g>0$ being the nondimensionalized gravity constant, and k being a positive multiplicative constant. If $v_n>v_c$, the system decompacts according to $\alpha(t)=-(1/2)kg(t-n)^2+v_n(t-n)+\alpha_{n-1}$ until the maximum decompaction $\alpha_{\max}=v_n^2/2kg+\alpha_{n-1}>\alpha_{n-1}$ has been reached at time $t=n+v_n/kg$. Then, the recompaction process $[\dot{\alpha}(t)<0]$ driven by gravity begins. To obtain a saturation of the recompaction at a finite positive value of α, an additional counteracting ''force'' that can be derived from a potential $V_n(\alpha)$ is reasonable. The physics of the problem suggests that the potential $V_n(\alpha)$ acts as a compaction barrier with $V_n(\alpha\to 0)\to\infty$ as the ultimate limit for compaction (corresponding to the closed-packed limit), and that the potential $V_n(\alpha)$ depends explicitly on the shake number or the history of the compaction process (''memory effect''). This is because $V_n(\alpha)$ originates from the internal resistance of the granular network against further compaction and this network changes and compacts from shake to shake. Therefore, the recompaction process can be modeled by

$$\ddot{\alpha}=-kg-\partial_\alpha V_n(\alpha) \quad \text{if } \dot{\alpha}<0 \tag{22}$$

and

$$\ddot{\alpha}=0 \quad \text{if } \dot{\alpha}=0. \tag{23}$$

Given an appropriate potential V_n as discussed above, Eq. (22) can be solved with the initial conditions $\alpha(t=n+v_n/kg)=\alpha_{\max}$ and $\dot{\alpha}(t=n+v_n/kg)=0$. The result is a decrease of $\alpha(t)$ that eventually reaches $\dot{\alpha}=0$ in finite time, then the recompaction stops. This new rest state defines the compaction ratio α_n. Due to the viscoplastic yield condition, the granular system stays at rest until the next sufficiently strong shake occurs [13].

So far, our considerations apply to periodic shaking processes where v_n is the same for all n, as well as to nonperiodic shaking processes where v_n can differ from shake to shake. What, however, is a good candidate for the potential V_n that leads to the map (3)? As in the case of decompaction, one can assume the recompaction is basically driven by the gravational term $-kg$ until the compaction ratio of the previous compacted rest state, α_{n-1}, has been reached again, so that $V_n(\alpha>\alpha_{n-1})=0$. At $\alpha(t)=\alpha_{n-1}$, the potential V_n is suddenly present and slows down further recompaction. Knowing the map (3), the form of the potential V_n can be determined. The result can be summarized as follows.

Proposition: Suppose that the decompaction and recompaction process (i) obeys the model Eqs. (21), (22), and (23) and (ii) is driven from rest, α_{n-1}, by periodic shaking, $v_n=\dot{\alpha}(t=n)=v>v_c$, for any n. If the potential $V_n(\alpha)$ possesses the structure

$$V_n(\alpha)=\begin{cases} 0 & \text{if } \alpha>\alpha_{n-1} \\ -kg\alpha+f_n\dfrac{1}{\alpha} & \text{if } \alpha\leqslant\alpha_{n-1}, \end{cases} \tag{24}$$

with $k>0$ and $f_n>0$ being a shake-dependent coefficient, the next compacted rest state α_n is given by the map (3) with $h_n=v^2/2f_n$.

Proof: First note that Eq. (22) has a first integral,

$$\tfrac{1}{2}\dot{\alpha}^2+kg\alpha+V_n(\alpha)=\text{const.} \tag{25}$$

One can take advantage of (25) by using that at time $t_r=n+2v/kg$, $\alpha(t_r)=\alpha_{n-1}$ and $\dot{\alpha}(t_r)=-v$ hold, and that the next rest state determines α_n. Inserting this into Eq. (24) yields

$$\frac{1}{2}v^2+f_n\frac{1}{\alpha_{n-1}}=f_n\frac{1}{\alpha_n} \tag{26}$$

or, equivalently,

$$\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}=\frac{v^2}{2f_n}. \tag{27}$$

Comparing Eq. (27) with Eq. (3) proves the proposition.

Note that the right-hand side of (27) is determined by the ratio of the initial condition of the nth shake, $v=\dot{\alpha}(n)$, which is a constant for periodic shaking, and the inverse of the shake-dependent steepness f_n of the potential $V_n(\alpha)$. By comparison with Eq. (5), one obtains

$$f_n=\frac{v^2}{2C}\left(1+\frac{n}{\nu}\right), \tag{28}$$

which implies that the steepness of the compaction barrier increases linearly from shake to shake for a $1/\ln(n)$ — compaction dynamics as observed in the experiment, Ref. [4].

The potential V_n in Eq. (24) consists of two parts: The first term being linear in α compensates gravitational recompaction whereas the second term accounts for the compaction barrier. The latter diverges proportionally to $1/\alpha$ and depends explicitly on the shake intensity $v=\dot{\alpha}(n)$ and the shake number n through the potential steepness f_n. Both dependences are not surprising; shake intensity and shake number influence directly the successive packing behavior of the granular system. Therefore, the shake dependence of f_n and also the inhomogeneity of the map (3) reflect the intrinsic compaction behavior of the shaken granular material. The arbitrariness of f_n in the potential $V_n(\alpha)$ can also lead to other types of compaction dynamics by choosing different shake dependence of f_n. In this context, it seems to be interesting if any granular system (even ultrafine powder) shows a $1/\ln(n)$ compaction dynamics.

No quantitative experimental results on the dynamics of the decompaction and recompaction process are available in the literature yet. Therefore, the qualitative model proposed above is speculative. Nevertheless, we think that it captures essential features of the process. Improving the model by taking into account collision effects of grains as well as nonhomogeneous corrections to the height dependence of the packing fraction require a microscopic approach based on the kinetic theory of inelastic many particle systems that is beyond the scope of this study.

III. DISCUSSION AND CONCLUSION

Introducing the compaction ratio α as an order parameter, we have given reasons for the conjecture that the nonautonomous map (3) might be an appropriate phenomenological description of the stroboscopic compaction dynamics of granular systems as studied in [3,4]. We have shown that this map with the special nonautonomous term h_n given in Eq. (4) or, equivalently, the stroboscopic decay law (5), leads to the empirical result of Knight *et al.* [3,4] for the successive compaction of the packing fraction. Based on Eq. (5), we have argued that — at least in principle — the decay behavior towards maximum compaction, $\alpha=0$, happens on two time scales. Finally, we have proposed a simple nonlinear model for the dynamics from shake to shake that is based on viscoplastic ideas and leads to the stroboscopic map (3).

The structural simplicity of the map (3) might indicate that it is universal in the sense that it holds for any type of granular system (ultrafine and fine powders, granules, and spheres as in the experiment [4]). It remains an open question whether all types of granular systems have the same h_n, Eq. (4), or the same shake dependence of the compaction barrier V_n as discussed in Sec. II F implying a logarithmic decay of the compaction ratio. We hope that our theory stimulates experiments that check the validity of our basic assumption, Eq. (5), as well as detailed experimental studies of the decompaction and recompaction process. For a recent alternative approach to the compaction problem, we refer to Ref. [14].

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[12] All three assumptions are only approximately valid. As far as (iii) is concerned, no significant ''liftoff'' of the grains on the bottom of the tall container has been observed (in contrast to granular convection experiments) [H. Jaeger (private communication)]. As far as assumption (i) is concerned, it is interesting that the packing fraction of the successive *compacted* states does not show a very significant vertical dependence. From Fig. 2 of Ref. [4] it follows that the relative difference of ρ_n near the top and bottom of the tall cylinder is about 2% for $\Gamma=1.8$ and basically zero for $\Gamma=2.7$.

[13] If the next shake occurs before the system returned to the rest state again, the granular system is in permanent motion leading to vibration-induced convection. In this case, however, pattern forming structures develop that are far beyond our model for compaction.

[14] E. Ben Naim, J.B. Knight, and E.R. Nowak (unpublished).